Two-scale Dirichlet-Neumann preconditioners for elastic problems with boundary refinements

Patrice Hauret [a] and Patrick Le Tallec [b,*]

[a] *Graduate Aeronautical Laboratories, MS 205-45, California Institute of Technology Pasadena, CA 91125, USA*

[b] *Laboratoire de Mécanique des Solides, CNRS UMR 7649, Département de Mécanique, Ecole Polytechnique, 91128 Palaiseau Cedex, FRANCE*

Abstract

The present work deals with the efficient resolution of elastostatics problems on domains presenting boundary refinements. The proposed approach consists in separating the boundary refinements from the interior of the domain by means of the mortar method, and in using Dirichlet-Neumann preconditioners to solve the corresponding algebraic system. We prove that the simplest Dirichlet-Neumann algorithm achieves the independence of the condition number of the preconditioned system with respect to the number and the size of the small details. Nevertheless, a degradation of the situation occurs when the refined boundary is clamped. An enhanced preconditioner is then designed by the introduction of a coarse space to mitigate the aforementioned sensitivity. Some numerical tests are performed to confirm the analysis, and the tools are extended by the proposition of a quasi-Newton method in the case of nonlinear elasticity. This paper is an extended version of a work presented at the DD16 conference with proofs and complete numerical results.

Key words: domain decomposition, Dirichlet-Neumann preconditioner, coarse grid, mortar methods

* Corresponding author.
Email addresses: `phauret@aero.caltech.edu` (Patrice Hauret), `patrick.letallec@polytechnique.fr` (Patrick Le Tallec).

1 Introduction

The present paper is devoted to the construction of efficient numerical procedures to solve vector elliptic problems with small geometric details on the boundary of the domain, where a *localized fine scale* behavior of the solution is expected. In particular, the solution in displacements $u \in \mathbb{R}^d$ of the linearized elastostatics problem will be considered, that is for $d = 2, 3$ the solution of

$$\begin{cases} -\, div(\mathbf{E} : \varepsilon(u)) = f \quad \text{ on } \Omega \subset \mathbb{R}^d, \\ u = 0 \quad \text{ on } \Gamma_D, \\ (\mathbf{E} : \varepsilon(u)) \cdot n = g \quad \text{ on } \Gamma_N. \end{cases} \tag{1}$$

Above, the linearized strain tensor is denoted by

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla^\top u\right),$$

and the fourth order tensor $\mathbf{E}$ is assumed to be elliptic over the set of symmetric matrices

$$\exists \alpha > 0, \forall \xi \in \mathbb{R}^{d\times d}, \xi^\top = \xi, \quad (\mathbf{E} : \xi) : \xi \geq \alpha\, \xi : \xi.$$

In our framework, we consider that inside the disjoint subsets $(\Omega_k)_{1\leq k\leq K}$ of Ω, the solution rapidly varies. In applications like tires development, one could think of geometric refinements or sculptures along the boundary $\partial\Omega$. At the opposite, the solution u has slow variations in the interior subdomain $\Omega_0 = \Omega \setminus \overline{(\cup_{1\leq k\leq K}\Omega_k)}$.

The strategy proposed in this paper consists in using a non-conforming mortar formulation for (1) in order to decompose the physical domain into coarse and fine zones. Then, Dirichlet-Neumann preconditioners are proposed, analysed and tested to solve the corresponding linear system for the approximate cost of inversion of the coarse system, that is the problem set over Ω_0. To do so, we assume that the computational cost of the local solution over each $(\Omega_k)_{1\leq k\leq K}$ is reasonably low when compared to the resolution over Ω_0. We then show for the proposed strategies a two-scale property, in the sense that the condition number of the preconditioned system remains independent of the number and the size of the small subdomains.

Mortar methods have been introduced for the first time in [BMP93,BMP94] as a weak coupling between subdomains with non-conforming meshes, or between subproblems solved with different approximation methods. The main purpose was to overcome the very sub-optimal "$\sqrt{h}$" error estimate obtained with

pointwise matching. The analysis of this method as a mixed formulation can be found in [Bel99]. For the present purpose, various Lagrange multipliers spaces can be indifferently adopted. For example, one can use the original formulation from [BMP93]. It is worth noticing that because of the disjoint character of the small subdomains, no modification of Lagrange multipliers is necessary on the boundary of the interfaces. Indeed, interfaces are only shared by two subdomains: the coarse one, and a fine one. Among other possibilities, let us quote the dual variant from [Woh00] which provides a diagonal constraint, or discontinuous stabilized formulations [ABFI99,Bel04,HT04a,HT04b] involving block-diagonal constraints. In the case of a second order approximation in displacements, one can also adopt the proposal from [Ses98], opting for affine Lagrange multipliers. The list is of course not exhaustive.
Moreover, the independence of the coercivity constant of the broken elastostatics bilinear form with respect to the number and the size of the subdomains has been proved in [Bre04,HT04a]. There is consequently no limitation in considering a high number of small subdomains, and error estimates remain optimal. A brief review of the non-conforming formulation adopted herein to discretize (1) is proposed in section 2.

The challenge is then to develop a solver which efficiently handles such situations. In the present framework, the disymmetric roles played by the coarse and fine subdomains privileges Dirichlet-Neumann preconditioners (see for instance [QV99,Woh01]), rather than symmetric strategies such as Neumann-Neumann [TRV91] or FETI [FR91], widely studied in the mortar setting [Tal93,AKP95,AMW99,AAKP99,Ste99]. In sections 3 and 4, we begin by proposing a basic Dirichlet-Neumann preconditioner and prove that its quality is independent of the number and of the size of the boundary refinements. In this sense, we can talk of two-scale preconditioning. Nevertheless, the quality of this first preconditioner deteriorates when a displacement boundary condition is imposed on a boundary refinement. This drawback is mitigated by the introduction of local coarse spaces lifting interface rigid motions. An enhanced Dirichlet-Neumann preconditioner insensitive to boundary conditions is then obtained and analyzed. These preconditioners are implemented in section 5 and tested in section 6 to confirm the theoretical analysis.

When considering nonlinear problems with soft geometric refinements on the boundary, it is illustrated in section 6 that such preconditioners can be used to build efficient quasi-Newton methods.

2 A mortar formulation

2.1 Continuous problem

Let $\Omega \subset \mathbb{R}^d$, be an open set partitioned into $K+1$ subsets $(\Omega_k)_{0\leq k\leq K}$ satisfying $\overline{\Omega} = \cup_{i=0}^K \overline{\Omega_k}$ and $\Omega_k \cap \Omega_l = \emptyset$ if $k, l \geq 1$. We denote by $\Gamma_{0k} = \overline{\Omega_0} \cap \overline{\Omega_k}$ the interface between Ω_0 and Ω_k, and the skeleton of the internal interfaces is denoted by $\mathcal{S} = \cup_{k=1}^K \Gamma_{0k}$. For the understanding of the situation, let us say that Ω_0 has slowly varying physical properties whereas the disjoint subsets $(\Omega_k)_{1\leq k\leq K}$ have rapidly varying ones or complex geometries. Moreover, the subdomain Ω_0 has a non-empty intersection with all the subdomains $(\Omega_k)_{1\leq k\leq K}$. We will also assume as a simplification that the intersection between two local subdomains Ω_k, $k \geq 1$ is empty. In other words, for the time being, the inclusions are disconnected. On the part Γ_D of the boundary $\partial\Omega$, an homogeneous Dirichlet boundary condition is imposed. Concerning the coefficients of the fourth order elasticity tensor $\mathbf{E}$, we assume that the stress tensor is symmetric whatever the deformation in the material, namely for almost all $x \in \Omega$,

$$\forall \xi \in \mathbb{R}^{d\times d}, \xi^\top = \xi, \quad \mathbf{E}(x) : \xi \text{ is a symmetric matrix.}$$

Moreover, the different materials are spectrally isotropic, namely for all $k \geq 1$, there exists two constants c_k and C_k, such that for almost all $x \in \Omega_k$,

$$\forall \xi \in \mathbb{R}^{d\times d}, \xi^\top = \xi, \quad c_k\, \xi : \xi \leq (\mathbf{E}(x) : \xi) : \xi \leq C_k\, \xi : \xi. \tag{2}$$

For homogeneous isotropic materials, if E_k stands for the Young modulus of the material used in Ω_k, both c_k and C_k are proportional to E_k.

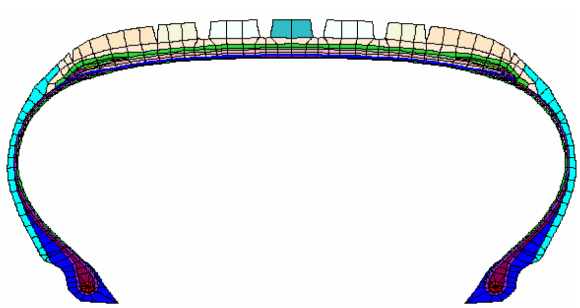

Fig. 1. Example of a structure presenting small geometric refinements on its boundary.

We introduce the following spaces for the displacement fields

$$H^1_*(\Omega) = \{v \in H^1(\Omega)^d,\ v|_{\Gamma_D} = 0\},$$

$$H^1_*(\Omega_k) = \{v \in H^1(\Omega_k)^d,\ v|_{\Gamma_D \cap \partial\Omega_k} = 0\},$$

$$X = \Big\{v \in L^2(\Omega)^d, \quad v_k = v|_{\Omega_k} \in H^1_*(\Omega_k), \forall k\Big\} = \prod_{k=0}^{K} H^1_*(\Omega_k),$$

X being endowed with the H^1 broken norm

$$\|v\|_X = \left(\sum_{k=0}^{K} \|v\|^2_{H^1(\Omega_k)^d}\right)^{\frac{1}{2}}.$$

We also introduce as interface space

$$M = \prod_{k=1}^{K} L^2(\Gamma_{0k})^d.$$

In the whole paper, for homogeneity reason, the H^1 norm is rescaled and defined by

$$\|v\|^2_{H^1(\Omega_k)^d} = \frac{1}{(L_k)^2}\|v\|^2_{L^2(\Omega_k)^d} + \|\nabla v\|^2_{L^2(\Omega_k)^d},$$

where L_k denotes the diameter of Ω_k.

Our elastodynamic problem (1) equivalently consists in finding $u \in H^1_*(\Omega)$ such that

$$a(u,v) = l(v), \quad \forall v \in H^1_*(\Omega), \tag{3}$$

where the continuous coercive bilinear form a is defined as

$$a(u,v) = \int_\Omega (\mathbf{E} : \varepsilon(u)) : \varepsilon(v), \quad \forall u, v \in H^1_*(\Omega),$$

and the continuous linear form l as

$$l(v) = \int_\Omega f \cdot v + \int_{\Gamma_N} g \cdot v, \quad \forall v \in H^1_*(\Omega),$$

with $f \in L^2(\Omega)^d$ and $g \in L^2(\Gamma_N)^d$. This problem is well-posed from Lax-Milgram lemma, by using Korn's inequality (see [DL72]) to prove the coercivity of the bilinear form a on $H^1_*(\Omega)$.

2.2 Discretization

We introduce here a domain-based non-conforming discretization of the problem using mortar elements. Well-posedness results and error estimates are then reviewed.

2.2.1 The mesh

For each $0 \leq k \leq K$, let us consider a family of shape regular meshes $(\mathcal{T}_{k;h_k})_{h_k>0}$ defined over each domain Ω_k, and denote $h_k = \sup_{T\in\mathcal{T}_{k;h_k}} diam(T)$ the maximal local diameter of the triangulation. The mesh $\mathcal{T}_{0;h_0}$ defined on Ω_0 is the coarsest, i.e $h_0 > h_k$ for all $1 \leq k \leq K$, and a non-conforming family of meshes $(\mathcal{T}_h)_{h>0}$ over Ω is obtained as $\mathcal{T}_h = \cup_{k=0}^K \mathcal{T}_{k,h_k}$, denoting $h = \max_{0\leq k\leq K} h_k$.

For each $1 \leq k \leq K$, the interface Γ_{0k} inherits from the family of meshes $(\mathcal{F}_{k;\delta_k})_{\delta_k>0}$ defined as the trace of the fine meshes $(\mathcal{T}_{k;h_k})_{h_k>0}$ over Γ_{0k}, with $\delta_k = \sup_{F\in\mathcal{F}_{k;\delta_k}} h(F)$. Moreover, each Γ_{0k} is assumed to be the union of faces of elements in $\mathcal{T}_{k;h_k}$ for any discretization. Such an assumption is in general referred to as *geometrical conformity.*

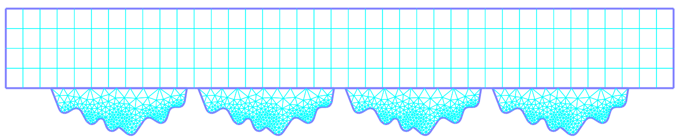

Fig. 2. Mesh of a 2D structure with boundary refinement. The interfaces between coarse and fine subdomains are geometrically compatible with the mesh of fine subdomains.

2.2.2 Mesh-dependent spaces

We define here some mesh-dependent spaces, endowed with useful mesh-dependent norms already proposed and used in [AT95,Woh99]. For each $1 \leq k \leq K$, they are defined as

$$\mathbb{H}_\delta^{1/2}(\Gamma_{0k}) = \{\phi \in L^2(\Gamma_{0k})^d, \|\phi\|^2_{\delta,\frac{1}{2},k} = \sum_{F\in\mathcal{F}_{k;\delta_k}} \frac{1}{h(F)}\|\phi\|^2_{L^2(F)^d} < +\infty\},$$

$$\mathbb{H}_\delta^{-1/2}(\Gamma_{0k}) = \{\lambda \in L^2(\Gamma_{0k})^d, \|\lambda\|^2_{\delta,-\frac{1}{2},k} = \sum_{F\in\mathcal{F}_{k;\delta_k}} h(F)\|\lambda\|^2_{L^2(F)^d} < +\infty\},$$

endowed respectively with the norms $\|\cdot\|_{\delta,\frac{1}{2},k}$ and $\|\cdot\|_{\delta,-\frac{1}{2},k}$. The product spaces $\mathbb{W}_\delta = \prod_{k=1}^K \mathbb{H}_\delta^{1/2}(\Gamma_{0k})$ and $\mathbb{M}_\delta = \prod_{k=1}^K \mathbb{H}_\delta^{-1/2}(\Gamma_{0k})$, are consequently provided with the product norms

$$\|\phi\|_{\delta,\frac{1}{2}} = \left(\sum_{k=1}^K \|\phi\|^2_{\delta,\frac{1}{2},k}\right)^{1/2}, \quad \|\lambda\|_{\delta,-\frac{1}{2}} = \left(\sum_{k=1}^K \|\lambda\|^2_{\delta,-\frac{1}{2},k}\right)^{1/2},$$

and can be seen as dual spaces by means of the L^2 inner product.

Remark 1 *The use of such mesh-dependent spaces instead of* $H_{00}^{1/2}(\Gamma_{0k})^d$ *and its dual* $H^{-1/2}(\Gamma_{0k})^d = \left(H_{00}^{1/2}(\Gamma_{0k})^d\right)'$ *has several advantages. First, these* *mesh-dependent norms are computable, which make easier a posteriori estimates (see [Woh99]) and penalized formulations (see [Hau04]). Moreover, such mesh dependent norms are at present time privileged tools for error estimates in the 3D setting.*

2.2.3 Non-conforming approximation

Let us introduce the discrete subspaces of degree q inside each subdomain

$$X_{k;h_k} = \{p \in H^1_*(\Omega_k) \cap \mathcal{C}^0(\Omega_k)^d, \quad p|_T \in \mathcal{P}_q(T), \forall T \in \mathcal{T}_{k;h_k}\} \oplus \mathcal{B}_{k;h_k},$$

with $\mathcal{P}_q = [\mathbb{P}_q]^d$ or $[\mathbb{Q}_q]^d$, where $\mathbb{P}_q$ (resp. $\mathbb{Q}_q$) is the space of polynomials of total (resp. partial) degree q, and where we have introduced a possible stabilization space $\mathcal{B}_{k;h_k}$ of interface bubbles as in [BM00,Bel04,Hau04]. The corresponding product space is denoted by $X_h = \prod_{k=0}^{K} X_{k;h_k} \subset X$. Moreover, let us define the following trace spaces on the non-mortar side (small subdomain side herein):

$$W_{k;\delta_k} = \{p|_{\Gamma_{0k}}, p \in X_{k;h_k}\}, \qquad W^0_{k;\delta_k} = W_{k;\delta_k} \cap H^1_0(\Gamma_{0k})^d,$$

endowed with the mesh-dependent norm $\|\cdot\|_{\delta,\frac{1}{2},k}$.

In order to formulate the weak continuity constraint, we introduce the spaces $M_{k;\delta_k}$ of (possibly discontinuous) Lagrange multipliers defined on the meshes $\mathcal{F}_{k;\delta_k}$. Achieving optimal approximation requires that such spaces contain all polynomials $[\mathbb{P}_{q-1}]^d$ of degree $q-1$. The product space $M_\delta = \prod_{k=1}^{K} M_{k;\delta_k}$ is endowed with the mesh-dependent norm $\|\cdot\|_{\delta,-\frac{1}{2}}$. Making use of the bilinear form

$$b(v,\lambda) = \sum_{k=1}^{K} \int_{\Gamma_{0k}} [v]\cdot\lambda, \qquad \forall (v,\lambda) \in X \times M$$

with the jump function $[v] = v_0 - v_k$ on the interfaces Γ_{0k}, we introduce the decomposition

$$\begin{aligned} b(v,\lambda) &= \sum_{k=1}^{K} \int_{\Gamma_{0k}} v_0 \cdot \lambda_k - \sum_{k=1}^{K} \int_{\Gamma_{0k}} v_k \cdot \lambda_k \\ &:= \sum_{k=1}^{K} b_{0k}(v_0,\lambda_k) - \sum_{k=1}^{K} b_k(v_k,\lambda_k), \end{aligned}$$

and the constrained space of admissible displacements

$$V_h = \{u_h \in X_h, \quad b(u_h,\lambda_h) = 0, \quad \forall \lambda_h \in M_\delta\}.$$

They are continuous "in average" across the interfaces $(\Gamma_{0k})_{1\le k\le K}$. Then, the problem of interest uses the broken elliptic form

$$\tilde{a}(u,v) := \sum_{k=0}^{K} a_k(u_k,v_k) = \sum_{k=0}^{K} \int_{\Omega_k} (\mathbf{E} : \varepsilon(u)) : \varepsilon(v), \qquad \forall u,v \in X,$$

and consists in finding $(u_h, \lambda_h) \in X_h \times M_\delta$, such that:

$$\begin{cases} \tilde{a}(u_h, v_h) + b(v_h, \lambda_h) = l(v_h), & \forall v_h \in X_h, \\ b(u_h, \mu_h) = 0, & \forall \mu_h \in M_\delta. \end{cases} \tag{4}$$

In other words, we solve our variational problem on the product space X_h under the kinematic continuity constraint $b(u_h, \cdot) = 0$.

2.2.4 Fundamental assumptions and error estimates

In order to ensure the well-posedness of the problem (4), some fundamental assumptions have to be made. Concerning the compatibility of X_h and M_δ, we assume (cf. [Woh01,Hau04]):

Assumption 1 *For each $1 \leq k \leq K$, there exists an operator:*

$$\pi_k : \mathbb{H}_\delta^{1/2}(\Gamma_{0k}) \to W_{k;\delta_k},$$

such that for all $v \in \mathbb{H}_\delta^{1/2}(\Gamma_{0k})$:

$$\int_{\Gamma_{0k}} (\pi_k v) \cdot \mu = \int_{\Gamma_{0k}} v \cdot \mu, \quad \forall \mu \in M_{k;\delta_k},$$

with:

$$\|\pi_k v\|_{\delta,\frac{1}{2},k} \leq C \|v\|_{\delta,\frac{1}{2},k}.$$

This assumption means that the projection perpendicular to the multiplier space onto the trace space $W_{k;\delta_k}$ is continuous. This implies a limitation on the size of M_δ with respect to X_h. If more than two subdomains had a common intersection, the range $W_{k;\delta_k}$ of π_k in assumption 1 would be replaced by $W^0_{k;\delta_k}$, in order to enable independent projections on each interface.

The coercivity of $\tilde{a}$ over $V_h \times V_h$ is obtained under the following assumption (cf. [Woh01,Hau04]):

Assumption 2 *For all $1 \leq k \leq K$, we assume that there exists a subspace $\tilde{M}_k$ of the Lagrange multipliers space $M_{k;\delta_k}$ such that $\tilde{M}_k \subset M_{k;\delta_k}$ independently of the mesh size δ_k. Moreover, we assume that if $v \in X$ is locally a rigid motion over all the $(\Omega_k)_{k \geq 1}$ in the sense that*

$$\tilde{a}(v, w) = 0, \quad \forall w \in X,$$

and satisfies

$$\int_{\Gamma_{0k}} [v] \cdot \mu = 0, \quad \forall \mu \in \tilde{M}_k, \quad k = 1, .., K,$$

then $v = 0$.

Various pairs of spaces $X_h \times M_\delta$ can be chosen to satisfy assumptions 1 and 2.

- The initial formulation from [BMP93,BMP94] proposes discrete displacements of degree q without stabilization, i.e. $\mathcal{B}_{k;h_k} = \emptyset$, and continuous Lagrange multipliers of degree q. In our framework, no modification of the Lagrange multipliers is necessary on the boundaries of the interfaces $(\partial\Gamma_{0k})_{1\leq k\leq K}$ because they are disjoint. Therefore, with this choice, the trace of fine displacements coincide with Lagrange multipliers, that is $M_{k;\delta_k} = W_{k;\delta_k}$ for all $1 \leq k \leq K$.
- In order to make the mortar weak continuity constraint diagonal, one can adopt the dual Lagrange multipliers from Wohlmuth [Woh00], again without special treatment on the boundaries of the interfaces.
- As shown in [Ses98] for second order approximations of the displacements ($q \geq 2$), the formulation from [BMP93,BMP94] can be modified by using only continuous Lagrange multipliers of degree $q - 1$.
- Discrete displacements of degree q with a proper stabilization are compatible with discontinuous Lagrange multipliers of degree $q - 1$, as proved in [Hau04,Bel04] or in [BM00] for three-field formulations.

In this framework, we recall the following optimal approximation result (cf. [Woh01,Hau04]):

Proposition 1 *Under assumptions 1 and 2, the problem (4) is well-posed. Moreover, if* $u \in \prod_{k=0}^{K} H^{q+1}(\Omega_k)^d$ *is solution of (3) with* $(\mathbf{E} : \varepsilon(u)) \in \prod_{k=0}^{K} H^q(\Omega_k)^{d\times d}$ *in which* $q \geq 1$*, and* $(u_h, \lambda_h) \in X_h \times M_\delta$ *is solution of (4), the following error estimate hold:*

$$\|u - u_h\|_X + \|\lambda - \lambda_h\|_{\delta,-\frac{1}{2}} \leq C \left(\sum_{k=1}^{K} h_k^{2q} \, |u|^2_{q+1,\mathbf{E},\Omega_k} \right)^{1/2},$$

with

$$|u|^2_{q+1,\mathbf{E},\Omega_k} = |u|^2_{H^{q+1}(\Omega_k)^d} + \frac{1}{C_k^2} \|\mathbf{E} : \varepsilon(u)\|^2_{H^q(\Omega_k)^{d\times d}}.$$

We have denoted the flux over the artificial interfaces by $\lambda = (\mathbf{E} : \varepsilon(u)) \cdot n$*, where the normal outward unit vector on* $\partial\Omega_0$ *is denoted by* n*.* C *denotes various constants independent of the decomposition into subdomains and of the discretization.*

Remark 2 (Choice of the non-mortar side) *In this discretization, as confirmed by assumption 1, we have chosen the non-mortar side defining the multipliers as the fine scale side of the interface* $\mathcal{S}$*. The main motivation is that* *in the preconditioners to be defined later, it is crucial to get a stable extension operator over the small scale subdomains, which is the case with the present choice.*

3 Two-scale preconditioners.

3.1 Notation

The previous discretization leads to a well-posed linear discrete problem with optimal error estimates. In this section, we propose and analyze preconditioners to solve this linear system for the approximate computational cost of the coarse scale problem on Ω_0, provided the solution of the local problem over each $(\Omega_k)_{1\leq k\leq K}$ be at a reasonnably low-cost. That is why we have assumed that the $(\Omega_k)_{1\leq k\leq K}$ were small and disjoint. Then, the inversions of the fine scale problems on the boundary can be parallelized and are relatively cheap in terms of computation.

Some notation and remarks must first be introduced:

- In this section, all quantities live in finite dimensional spaces. If a is a bilinear form, then $\mathbf{A}$ represents the matrix of a in the discrete space. If u is a function, then U is the vector of its nodal degrees of freedom in the chosen discrete space.
- For all $0 \leq k \leq K$, the bilinear form $a_k(\cdot,\cdot)$ is continuous in $H^1(\Omega_k)^d \times H^1(\Omega_k)^d$ and its continuity constant is C_k, already defined in (2).
- When $\Gamma_D \cap \partial\Omega_0$ has a positive measure, $a_0(\cdot,\cdot)$ is coercive in $H^1_*(\Omega_0) \times H^1_*(\Omega_0)$. We denote by α_0 its constant of coercivity, which is proportional to c_0 defined in (2), within a shape dependent constant.
- For all $1 \leq k \leq K$ such that Ω_k is fixed on a part of its boundary, the bilinear form $a_k(\cdot,\cdot)$ is coercive over $H^1_*(\Omega_k) \times H^1_*(\Omega_k)$ and its coercivity constant is denoted by α_k. It is proportional to c_k defined in (2), within a constant which depends continuously on the shape of Ω_k but not of its size because a_k and the scaled norm of H^1 have the same dependence with respect to a change of scale.

With obvious notation, the discrete problem (4) leads to the following linear system to solve

$$\begin{cases} \mathbf{A}_0 U_0 + \sum_{k=1}^{K} \mathbf{B}_{0k}^\top \Lambda_k = F_0, \\ \mathbf{A}_k U_k - \mathbf{B}_k^\top \Lambda_k = F_k, \quad 1 \leq k \leq K, \\ \mathbf{B}_{0k} U_0 - \mathbf{B}_k U_k = 0, \quad 1 \leq k \leq K. \end{cases} \tag{5}$$

Defining the local extended stiffness matrix of the k-th ($k \geq 1$) subproblem by

$$\mathbf{K}_k = \begin{pmatrix} \mathbf{A}_k & -\mathbf{B}_k^\top \\ -\mathbf{B}_k & 0 \end{pmatrix},$$

the problem (5) can be rewritten as

$$\begin{cases} \mathbf{A}_0 U_0 + \sum_{k=1}^{K} \mathbf{B}_{0k}^\top \Lambda_k = F_0, \\ \mathbf{K}_k \begin{pmatrix} U_k \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} F_k \\ -\mathbf{B}_{0k} U_0 \end{pmatrix}, \quad 1 \leq k \leq K. \end{cases} \tag{6}$$

The algebraic elimination of Λ_k in (6) leads to

$$\begin{cases} \mathbf{D}_0 U_0 = \overline{F_0}, \\ \mathbf{K}_k \begin{pmatrix} U_k \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} F_k \\ -\mathbf{B}_{0k} U_0 \end{pmatrix}, \quad 1 \leq k \leq K. \end{cases} \tag{7}$$

Denoting by R_k the canonical restriction from $X_{k;h_k} \times M_{k;\delta_k}$ to $M_{k;\delta_k}$ whose matrix is $\left(0, I_{M_{k;\delta_k}}\right)$, we have introduced the Schur complement matrix

$$\mathbf{D}_0 = \mathbf{A}_0 - \sum_{k=1}^{K} \mathbf{B}_{0k}^\top R_k \mathbf{K}_k^{-1} R_k^\top \mathbf{B}_{0k}$$

resulting from the elimination of small scales, and the equivalent coarse force

$$\overline{F_0} := F_0 - \sum_{k=1}^{K} \mathbf{B}_{0k}^\top R_k \mathbf{K}_k^{-1} \left(F_k, 0\right)^\top.$$

The problem is now split into a coarse problem defined on Ω_0, and into fine problems defined on $(\Omega_k)_{1 \leq k \leq K}$ using the coarse solution U_0. Of course, the elimination of small scales is still hidden in the definition of $\mathbf{D}_0$ and making the splitting of scales effective imposes an iterative resolution of (7) where $\mathbf{D}_0^{-1}$ will be replaced by an approximate inverse $\hat{\mathbf{D}}_0^{-1}$.

3.3 Two possible definitions for $\hat{\mathbf{D}}_0$

3.3.1 A symmetrized Dirichlet-Neumann preconditioner

The simplest idea consists in replacing the Schur complement $\mathbf{D}_0$ by the stiffness of the coarse problem $\hat{\mathbf{D}}_0 = \mathbf{A}_0$, which reduces the proposed preconditioning to a symmetrized Dirichlet-Neumann iteration. Indeed, the resolution of the approximate solution requires to solve

(1) Dirichlet problems on the $(\Omega_k)_{1\le k\le K}$ with zero weak trace on the interfaces $(\Gamma_{0k})_{1\le k\le K}$ to obtain the resultant interface forces and therefore $\overline{F_0}$,
(2) a Neumann problem on Ω_0 with the sollicitation $\overline{F_0}$ to compute U_0,
(3) Dirichlet problems on the $(\Omega_k)_{1\le k\le K}$ to compute the $(U_k)_{1\le k\le K}$ with forces $(F_k)_{1\le k\le K}$ and U_0 as a weak trace over the interfaces $(\Gamma_{0k})_{1\le k\le K}$.

Observe that steps 1 and 3 are compatible with a parallel implementation. In section 4.2, we shall prove that the condition number of the associated preconditioned system is independent of the number and of the size of the fine scale subdomains $(\Omega_k)_{k\ge1}$. We also prove that the method is efficient as soon as the ratio of stiffnesses between Ω_0 and the $(\Omega_k)_{k\ge1}$ does not become small, and when the small subdomains are not fixed on a part of their boundary.

3.3.2 An enhanced symmetrized Dirichlet-Neumannn preconditioner

The previous simplest choice of preconditioner may lack of efficiency when

- the substructure Ω_k is of small size and is fixed on a part of its boundary. In this situation, because of its size, the substructure will have a rather large stiffness to interface rigid body displacements.
- the substructure Ω_k may have other privileged directions of large stiffness to interface motions (rigid links, incompressibility).

Assuming that these directions of interface localized stiffness be in very small number N_k (this is indeed the case for interface rigid body motions), we propose a modification of the previous preconditioner enabling to correct such a lack of efficiency.

For all $k \ge 1$ such that Ω_k is fixed on a part of its boundary, we denote by $(e_k^i)_{1\le i\le N_k}$ (with $N_k = 6$ in general) the interface rigid motions of Γ_{0k} or rigid links and introduce $\overset{\circ}{W}_k = span\{e_k^i, i = 1, .., N_k\}$. To each interface rigid body motion e_k^i, we associate its local a_k-harmonic extension $(u_k^i, \lambda_k^i) \in X_{k;h_k} \times M_{k;\delta_k}$ solution of

$$\begin{cases} a_k(v, u_k^i) - \int_{\Gamma_{0k}} v \cdot \lambda_k^i = 0, \quad \forall v \in X_{k;h_k}, \\ -\int_{\Gamma_{0k}} u_k^i \cdot \mu = -\int_{\Gamma_{0k}} e_k^i \cdot \mu, \quad \forall \mu \in M_{k;\delta_k}. \end{cases} \tag{8}$$

These solutions span two small local spaces

$$\overset{\circ}{X}_k = span\{u_k^i, i = 1, .., N_k\} \subset X_{k;h_k}, \quad \overset{\circ}{M}_k = span\{\lambda_k^i, i = 1, .., N_k\} \subset M_{k;\delta_k}.$$

If $k \geq 1$ is such that Ω_k is not fixed on its boundary, the convention that $\overset{\circ}{W}_k = \overset{\circ}{M}_k = \{0\}$ and $N_k = 0$ is adopted. Instead of finding U_0 such that $\mathbf{D}_0 U_0 = \overline{F_0}$, we propose to compute $u_0 \in X_{0;h_0}$, $(u_k) \in (\overset{\circ}{X}_k)_{1 \leq k \leq K}$, $(\lambda_k) \in (\overset{\circ}{M}_k)_{1 \leq k \leq K}$ solution of the coupled problem

$$\begin{cases} a_0(u_0, v_0) + \sum_{k=1}^{K} \int_{\Gamma_{0k}} v_0 \cdot \lambda_k = \overline{l_0}(v_0), \quad \forall v_0 \in X_{0;h_0}, \\ a_k(u_k, v_k) - \int_{\Gamma_{0k}} v_k \cdot \lambda_k = 0, \quad \forall v_k \in \overset{\circ}{X}_k, \quad 1 \leq k \leq K, \\ -\int_{\Gamma_{0k}} u_k \cdot \mu_k = -\int_{\Gamma_{0k}} u_0 \cdot \mu_k, \quad \forall \mu_k \in \overset{\circ}{M}_k, \quad 1 \leq k \leq K, \end{cases} \tag{9}$$

where $\overline{l_0}$ is the linear form associated to the coarse sollicitation $\overline{F_0}$.

We introduce the matrix $\mathbf{I}_{0k} \in \mathbb{R}^{N_k \times dim X_{0;h_0}}$ defined for all $v_0 \in X_{0;h_0}$ by

$$(\mathbf{I}_{0k} V_0)_i = \int_{\Gamma_{0k}} v_0 \cdot \lambda_k^i = \left\langle \mathbf{B}_{0k} V_0, \Lambda_k^i \right\rangle, \quad \forall i = 1, .., N_k,$$

that is $\mathbf{I}_{0k} = \left[\Lambda_k^1, .., \Lambda_k^{N_k}\right]^\top \mathbf{B}_{0k} = \mathbf{\Lambda}_k^\top \mathbf{B}_{0k}$, and the restriction $\overset{\circ}{\mathbf{A}}_k$ of the displacement stiffness matrix $\mathbf{A}_k$ on the local space $\overset{\circ}{X}_k$. Thus

$$\left(\overset{\circ}{\mathbf{A}}_k\right)_{ij} = (U_k^i)^\top \mathbf{A}_k U_k^j = a_k(u_k^j, u_k^i) = \int_{\Gamma_{0k}} u_k^j \cdot \lambda_k^i,$$

and keeping (8)-1 in mind, the system (9) reads

$$\begin{cases} \mathbf{A}_0 U_0 + \sum_{k=1}^K \mathbf{I}_{0k}^\top \Theta_k = \overline{F_0}, \\ \overset{\circ}{\mathbf{A}}_k Z_k - \overset{\circ}{\mathbf{A}}_k^\top \Theta_k = 0, \\ -\overset{\circ}{\mathbf{A}}_k Z_k = -\mathbf{I}_{0k} U_0, \quad 1 \leq k \leq K. \end{cases} \tag{10}$$

The new vector Θ_k (resp. Z_k) denotes the component of λ_k (resp. u_k) in $\overset{\circ}{M}_k$ (resp. $\overset{\circ}{W}_k$) in (9). From the elimination of Θ_k and Z_k in (10), one obtains that $\hat{\mathbf{D}}_0 U_0 = \overline{F_0}$, with the new approximate Schur complement given by

$$
\begin{aligned}
\hat{\mathbf{D}}_0 &= \mathbf{A}_0 + \sum_{k=1}^{K} \mathbf{I}_{0k}^{\top} \mathring{\mathbf{A}}_k^{-t} \mathbf{I}_{0k} \\
&= \mathbf{A}_0 + \sum_{k=1}^{K} \mathbf{B}_{0k}^{\top} \boldsymbol{\Lambda}_k \mathring{\mathbf{A}}_k^{-t} \boldsymbol{\Lambda}_k^{\top} \mathbf{B}_{0k}.
\end{aligned} \tag{11}
$$

Its complexity is much smaller than (7) because the local problem (10)-2,(10)-3 for the subproblem $k \geq 1$ used in the construction of $\hat{\mathbf{D}}_0$, is only of dimension $N_k = 6$.

For analysis purpose, let us observe that this enhanced Dirichlet-Neumann preconditioner corresponds to a Dirichlet-Neumann decomposition where the Dirichlet substructures are defined by

$$
\mathring{X}_{k;h_k}^{\perp} = \{u_k \in X_{k;h_k}, \quad \int_{\Gamma_{0k}} u_k \cdot \mu = 0, \quad \forall \mu \in \mathring{M}_k\}, \quad 1 \leq k \leq K,
$$

and the Neumann substructure by

$$
\mathring{X}_h = \{u \in X_h, \quad b(u, \mu) = 0, \quad \forall \mu \in \mathring{M}_k\}.
$$

The analysis of this preconditioner is performed in section 4.3, proving now an independence with respect to essential boundary conditions imposed over the small subdomains $(\Omega_k)_{k \geq 1}$.

In order to state a preliminary remark, the following definition is required.

Definition 1 *For any $v_0 \in X_{0;h_0}$, its "rigid body projection" over Ω_k denoted by $\mathring{\pi}_k v_0 \in \mathring{X}_k$ is defined as the solution of (10)-2,(10)-3 for the subproblem k. More precisely $(\mathring{\pi}_k v_0, \mathring{\lambda}_k) \in \mathring{X}_k \times \mathring{M}_k$ is such that:*

$$
\begin{cases}
a_k(\mathring{\pi}_k v_0, v_k) - \displaystyle\int_{\Gamma_{0k}} \mathring{\lambda}_k \cdot v_k = 0, \quad \forall v_k \in \mathring{X}_k, \\
- \displaystyle\int_{\Gamma_{0k}} \mathring{\pi}_k v_0 \cdot \mu_k = - \int_{\Gamma_{0k}} v_0 \cdot \mu_k, \quad \forall \mu_k \in \mathring{M}_k.
\end{cases} \tag{12}
$$

In matricial form, we have $\mathring{\pi}_k v_k = \sum_{j=1}^{N_k} z_j u_k^j$ with $-\mathring{\mathbf{A}}_k Z = -\mathbf{I}_{0k} V_0$, yielding

$$
\mathring{\Pi}_k V_0 = \left[U_k^1, .., U_k^{N_k}\right] \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} V_0 = \mathring{\mathbf{U}}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} V_0,
$$

or equivalently $\mathring{\Pi}_k = \mathring{\mathbf{U}}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k}$. By construction of $\mathring{\mathbf{A}}_k$, one is now able to deduce that

$$
\begin{aligned}
\mathring{\Pi}_k^{\top} \mathbf{A}_k \mathring{\Pi}_k &= \mathbf{I}_{0k}^{\top} \mathring{\mathbf{A}}_k^{-t} \mathring{\mathbf{U}}_k^{\top} \mathbf{A}_k \mathring{\mathbf{U}}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} \\
&= \mathbf{I}_{0k}^{\top} \mathring{\mathbf{A}}_k^{-t} \mathring{\mathbf{A}}_k \mathring{\mathbf{A}}_k^{-1} \mathbf{I}_{0k} \\
&= \mathbf{I}_{0k}^{\top} \mathring{\mathbf{A}}_k^{-t} \mathbf{I}_{0k},
\end{aligned}
$$

and therefore the new preconditioner (11) takes the form

$$\hat{\mathbf{D}}_0 = \mathbf{A}_0 + \sum_{k=1}^{K} \mathring{\Pi}_k^\top \mathbf{A}_k \mathring{\Pi}_k. \tag{13}$$

Also observe from (8) that when a_k is symmetric,

$$\mathring{\pi}_k e_k^i = u_k^i, \quad 1 \le i \le N_k. \tag{14}$$

4 Condition number analysis

In this section, we establish upper bounds for the condition number of the preconditioned systems when using the two symmetrized Dirichlet-Neumann preconditioners defined in the subsections 3.3.1 and 3.3.2 respectively. First, the same factorized form for the original linear system and the preconditioner is introduced. Then, we show the spectral equivalence between $\hat{\mathbf{D}}_0$ and $\mathbf{D}_0$, detailing the dependence of the constants on the size of the domains, the stiffness of the materials, and on the mesh sizes. We finally deduce estimates on the condition number of the preconditioned system.

4.1 Factorization

The original system to solve is $\mathbf{A}(U_0, U_1, \Lambda_1, ..., U_K, \Lambda_K)^\top = (F_0, F_1, 0, ..., F_K, 0)^\top$, under the notation

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_0 & 0 & \mathbf{B}_{01}^\top & \dots & 0 & \mathbf{B}_{0K}^\top \\ 0 & \mathbf{A}_1 & -\mathbf{B}_1^\top & & & \\ \mathbf{B}_{01} & -\mathbf{B}_1 & 0 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & \mathbf{A}_K & -\mathbf{B}_K^\top \\ \mathbf{B}_{0K} & & & & -\mathbf{B}_K & 0 \end{pmatrix}.$$

Introducing the triangular and block diagonal matrices

$$T = \begin{pmatrix} I & 0 \dots 0 \\ \mathbf{K}_1^{-1} R_1^\top \mathbf{B}_{01} & I \\ \vdots & \ddots \\ \mathbf{K}_K^{-1} R_K^\top \mathbf{B}_{0K} & 0 \dots I \end{pmatrix}, \qquad H = \begin{pmatrix} \mathbf{D}_0 & 0 & \dots & 0 \\ 0 & \mathbf{K}_1 & & \\ \vdots & & \ddots & \\ 0 & & & \mathbf{K}_K \end{pmatrix},$$

a simple calculus shows that the factorization $\mathbf{A} = T^\top H T$ holds.

The matrix of our preconditioners can be written under the similar form $\mathbf{C} = T^\top \hat{H} T$, where $\hat{H}$ is the block diagonal matrix obtained when replacing $\mathbf{D}_0$ by $\hat{\mathbf{D}}_0$ in the definition of H. The approximate solution is then defined by

$$\mathbf{C}(\tilde{U}_0, \tilde{U}_1, \tilde{\Lambda}_1, ..., \tilde{U}_K, \tilde{\Lambda}_K)^\top = (F_0, F_1, 0, ..., F_K, 0)^\top.$$

Observe that both matrices $\mathbf{A}$ and $\mathbf{C}$ remains definite when discrete unknowns belong to the space of weakly continuous displacements

$$E = \{U = (U_0, U_1, \Lambda_1, .., U_K, \Lambda_K)^\top; \mathbf{B}_{0k} U_0 = \mathbf{B}_k U_k, 1 \leq k \leq K\}.$$

The sequel is devoted to the derivation of upper bounds for the condition number $\kappa_{\mathbf{A},E}(\mathbf{C}^{-1}\mathbf{A})$ in $\mathbf{A}$-norm over E.

4.2 Spectral equivalence for the simple Dirichlet-Neumann

We show herein the spectral equivalence between the Schur complement $\mathbf{D}_0$ and its approximation $\hat{\mathbf{D}}_0 = \mathbf{A}_0$ for the symmetrized Dirichlet-Neumann preconditioner presented in subsection 3.3.1. We prove:

Proposition 2 *Assuming that $\mathbf{A}_0$ is invertible that is $\Gamma_D \cap \partial\Omega_0$ has a positive measure, the following spectral equivalence holds for all U_0:*

$$W_{1,h} \langle \mathbf{D}_0 U_0, U_0 \rangle \leq \langle \mathbf{A}_0 U_0, U_0 \rangle \leq \langle \mathbf{D}_0 U_0, U_0 \rangle,$$

with

$$\frac{1}{W_{1,h}} = 1 + C \left(\max_{k \in I_1} \frac{C_k}{c_0} + \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} \right),$$

where I_1 (resp. I_2) is the set of indices $k \geq 1$ such that Ω_k is not fixed on its boundary (resp. is fixed on a part of its boundary). The constant C is independent of the number K and the size of the subdomains.

Observe that the condition number deteriorates for a small fixed subdomain $L_k << L_0$, $k \in I_2$, and for very stiff subdomains $C_k >> \alpha_0$.

Let us recall first the following lemma (see for instance [Bré99], page 158).

Lemma 1 *Let us assume that Γ_{0k} is of class $\mathcal{C}^1$. Then, there exists an open set $\Omega'_k \subset \Omega_0$ which is the restriction of a neighborhood of Γ_{0k} to Ω_0, and a linear extension operator $\mathcal{D}_k : H^1(\Omega'_k)^d \to H^1(\Omega_k)^d$ such that for all $u \in H^1(\Omega_0)^d$,*

$\mathcal{D}_k u = u$ on Γ_{0k} and

$$\int_{\Omega_k} (\mathcal{D}_k u)^2 \leq C \int_{\Omega'_k} u^2, \qquad \int_{\Omega_k} (\nabla \mathcal{D}_k u)^2 \leq C \int_{\Omega'_k} (\nabla u)^2,$$

where the constant C does not depend on Ω_k.

Proof : [of the proposition] Let U_0 be given. For all $k \geq 1$, let us define (U_k, Λ_k) defining a local harmonic lifting of U_0, obeying

$$\begin{pmatrix} \mathbf{A}_k & -\mathbf{B}_k^\top \\ -\mathbf{B}_k & 0 \end{pmatrix} \begin{pmatrix} U_k \\ \Lambda_k \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{B}_{0k} U_0 \end{pmatrix}. \tag{15}$$

In other words, we have $\Lambda_k = -R_k \mathbf{K}_k^{-1} R_k^\top \mathbf{B}_{0k} U_0$ and then by construction of U_k and Λ_k

$$-\left\langle \mathbf{B}_{0k}^\top R_k \mathbf{K}_k^{-1} R_k^\top \mathbf{B}_{0k} U_0, U_0 \right\rangle = \left\langle \mathbf{B}_{0k}^\top \Lambda_k, U_0 \right\rangle = \langle \Lambda_k, \mathbf{B}_k U_k \rangle = \langle \mathbf{A}_k U_k, U_k \rangle \geq 0.$$

Adding $\langle \mathbf{A}_0 U_0, U_0 \rangle$ on both sides of the inequality proves that $\langle \mathbf{D}_0 U_0, U_0 \rangle \geq \langle \mathbf{A}_0 U_0, U_0 \rangle$.

Let us now bound $\mathbf{A}_0$ from below. Let $u_0 \in H^1_*(\Omega)$ be given. For all $k \geq 1$ such that Ω_k has an empty intersection with Γ_D (we denote $k \in I_1$), we decompose u_0 on Ω'_k (as defined in lemma 1) into $u_0 = r_k + w_k$ in Ω'_k, where r_k belongs to the space $\mathcal{R}(\Omega'_k)$ of rigid motions over Ω'_k and

$$\int_{\Omega'_k} w_k \cdot r = 0, \quad \forall r \in \mathcal{R}(\Omega'_k). \tag{16}$$

We define the displacements $u_k = r_k + u'_k$ in Ω_k where $r_k \in \mathcal{R}(\Omega_k)$ is the natural extension to Ω_k of $r_k \in \mathcal{R}(\Omega'_k)$ (thus $r_k \in \mathcal{R}(\Omega_k \cup \Omega'_k)$), and

$$u'_k = \mathcal{I}_{k;h_k} \mathcal{D}_k w_k + \mathcal{R}_{k;\delta_k} \pi_k (w_k - \mathcal{I}_{k;h_k} \mathcal{D}_k w_k),$$

where $\mathcal{I}_{k;h_k}$ denotes the Scott-Zhang [SZ90] interpolation over $X_{k;h_k}$, and $\mathcal{R}_{k;\delta_k}$ is the extension by zero operator over the grid points of Ω_k. By construction, the mortar condition holds

$$\int_{\Gamma_{0k}} u_k \cdot \mu = \int_{\Gamma_{0k}} u_0 \cdot \mu, \quad \forall \mu \in M_{k;\delta_k}.$$

Moreover, by using the stability of the extension operator $\mathcal{R}_{k;\delta_k}$ from $W_{k;\delta_k}$ to $H^1(\Omega_k)^d$, the assumption 1, the stability of $\mathcal{I}_{k;h_k}$ from $H^1(\Omega_k)^d$ to $H^1(\Omega_k)^d$, the classical estimate (see [SZ90]) $\|u - \mathcal{I}_{k;h_k} u\|_{\delta,\frac{1}{2},k} \leq C|u|_{H^1(\Omega_k)^d}$, and the stability property of $\mathcal{D}_k$ from lemma 1, we obtain

$$
\begin{aligned}
& a_k(u_k, u_k) \\
& \leq C_k \int_{\Omega_k} |\nabla u_k'|^2 = C_k \, |u_k'|^2_{H^1(\Omega_k)^d} \\
& \leq 2C_k \, |\mathcal{I}_{k;h_k} \mathcal{D}_k w_k|^2_{H^1(\Omega_k)^d} + 2C_k \, |\mathcal{R}_{k;\delta_k} \pi_k (\mathcal{D}_k w_k - \mathcal{I}_{k;h_k} \mathcal{D}_k w_k)|^2_{H^1(\Omega_k)^d} \\
& \leq 2C_k \, |\mathcal{I}_{k;h_k} \mathcal{D}_k w_k|^2_{H^1(\Omega_k)^d} + 2CC_k \|\pi_k (\mathcal{D}_k w_k - \mathcal{I}_{k;h_k} \mathcal{D}_k w_k)\|^2_{\delta,\frac{1}{2},k} \\
& \leq 2C_k \, |\mathcal{I}_{k;h_k} \mathcal{D}_k w_k|^2_{H^1(\Omega_k)^d} + 2CC_k \|\mathcal{D}_k w_k - \mathcal{I}_{k;h_k} \mathcal{D}_k w_k\|^2_{\delta,\frac{1}{2},k} \\
& \leq CC_k \, |\mathcal{D}_k w_k|^2_{H^1(\Omega_k)^d} \leq CC_k \, |w_k|^2_{H^1(\Omega_k')^d} \, . \qquad (17)
\end{aligned}
$$

Moreover, denoting by $\mathfrak{R}_k = \{r \in \mathcal{R}(\Omega_k'), \int_{\Omega_k'} r = 0, \|r\|_{L^2(\Omega_k')^d} = 1\}$, the following inequality holds for all $v \in H^1(\Omega_k')^d$

$$
|v|^2_{H^1(\Omega_k')^d} \leq C_{\Omega_k'} \left(\int_{\Omega_k'} \varepsilon(v) : \varepsilon(v) + \frac{1}{diam(\Omega_k')^2} \left(\sup_{r \in \mathfrak{R}_k} \int_{\Omega_k'} v \cdot r \right)^2 \right). \qquad (18)
$$

The constant $C_{\Omega_k'}$ is independent of the size of Ω_k' from the adopted scaling of the norms, but possibly depending on its shape. The shape independence of this constant is insured for polyhedral shape regular domains in [Bre04], or in [Hau04] for slightly less restrictive assumptions. Therefore, we have from (16) by definition of w_k that

$$
|w_k|^2_{H^1(\Omega_k')^d} \leq C_{\Omega_k'} \int_{\Omega_k'} \varepsilon(w_k) : \varepsilon(w_k).
$$

Keeping (17) in mind, a summation over $k \in I_1$ provides

$$
\sum_{k \in I_1} a_k(u_k, u_k) \leq C \sum_{k \in I_1} C_k \int_{\Omega_k'} \varepsilon(u_0) : \varepsilon(u_0), \qquad (19)
$$

with a constant C independent of the size of the subdomains. Since by construction $\cup_{k \in I_1} \Omega_k' \subset \Omega_0$, and since there is a bounded number of domains Ω_k' overlapping at a given point, we deduce

$$
\sum_{k \in I_1} a_k(u_k, u_k) \leq C \max_{k \in I_1}(C_k) \int_{\Omega_0} \varepsilon(u_0) : \varepsilon(u_0) \leq \frac{C}{c_0} \max_{k \in I_1}(C_k) a_0(u_0, u_0).
$$

For all $k \geq 1$ such that Γ_D is fixed on a part of its boundary (that is $k \in I_2$), we cannot use the extension operator $\mathcal{D}_k$ because it will not satisfy the Dirichlet boundary condition on Γ_D. But, the Sobolev lifting theorem proves the existence of a function $\tilde{u}_k$ whose trace is u_0 on Γ_{0k} and such that

$$
\frac{1}{(L_k)^2} \int_{\Omega_k} |\tilde{u}_k|^2 + \int_{\Omega_k} |\nabla \tilde{u}_k|^2 \leq C \left(\frac{1}{L_k} \int_{\Gamma_{0k}} \langle u_0 \rangle_k^2 + |u_0|^2_{H^{1/2}(\Gamma_{0k})^d} \right).
$$

Here, $\langle u_0 \rangle_k$ denotes the average $\langle u_0 \rangle_k = \frac{1}{meas(\Gamma_{0k})} \int_{\Gamma_{0k}} u_0$ of u_0 on Γ_{0k} and C is a constant which is independent of the size of Ω_k but which depends on the ratio between L_k and the distance from Γ_{0k} to Γ_D. We then modify $\tilde{u}_k$ to obtain a discrete function satisfying the weak-continuity constraint on Γ_{0k}, and define using our previous notation $u_k = \mathcal{I}_{k;h_k}\tilde{u}_k + \mathcal{R}_{k;\delta_k}\pi_k(\tilde{u}_k - \mathcal{I}_{k;h_k}\tilde{u}_k)$. By construction, the mortar condition is satisfied

$$\begin{aligned}\int_{\Gamma_{0k}} u_k \cdot \mu &= \int_{\Gamma_{0k}} (\mathcal{I}_{k;h_k}\tilde{u}_k + \tilde{u}_k - \mathcal{I}_{k;h_k}\tilde{u}_k) \cdot \mu \\ &= \int_{\Gamma_{0k}} u_0 \cdot \mu, \quad \forall \mu \in M_{k;\delta_k}.\end{aligned}$$

From the same argument as in the case $k \in I_1$, we get

$$\begin{aligned}a_k(u_k, u_k) &\leq C C_k \int_{\Omega_k} |\nabla \tilde{u}_k|^2 \\ &\leq C C_k \left(\frac{1}{L_k} \int_{\Gamma_{0k}} \langle u_0 \rangle_k^2 + |u_0|^2_{H^{1/2}(\Gamma_{0k})^d} \right) \\ &\leq C C_k \frac{L_0}{L_k} \left(\frac{1}{L_0} \int_{\Gamma_{0k}} \langle u_0 \rangle_k^2 + |u_0|^2_{H^1(\Omega'_k)^d} \right). \end{aligned} \tag{20}$$

Observe that Cauchy-Schwarz inequality provides

$$\begin{aligned}\sum_k \int_{\Gamma_{0k}} \langle u_0 \rangle_k^2 &= \sum_k meas(\Gamma_{0k}) \, \langle u_0 \rangle_k^2 = \sum_k meas(\Gamma_{0k})^{-1} \left(\int_{\Gamma_{0k}} 1 u_0 \right)^2 \\ &\leq \sum_k meas(\Gamma_{0k})^{-1} \int_{\Gamma_{0k}} u_0^2 \int_{\Gamma_{0k}} 1 \leq \sum_k \int_{\Gamma_{0k}} u_0^2 = \int_{\Gamma_0} u_0^2,\end{aligned}$$

and summing (20) for $k \in I_2$ implies

$$\begin{aligned}\sum_{k \in I_2} a_k(u_k, u_k) &\leq C \max_{k \in I_2} \left(C_k \frac{L_0}{L_k} \right) \left(\frac{1}{L_0} \int_{\Gamma_0} u_0^2 + |u_0|^2_{H^1(\partial\Omega_0)^d} \right) \\ &\leq C \max_{k \in I_2} \left(C_k \frac{L_0}{L_k} \right) \|u_0\|^2_{H^1(\Omega_0)^d} \\ &\leq C \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} \, a_0(u_0, u_0).\end{aligned}$$

As a consequence, with this choice of u_k

$$\langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k U_k, U_k \rangle \leq \left(1 + C \max_{k \in I_1} \frac{C_k}{c_0} + C \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} \right) \langle \mathbf{A}_0 U_0, U_0 \rangle .$$

To conclude the proof, let us show that for all $(V_k)_{k \geq 1}$ such that $\mathbf{B}_k V_k = \mathbf{B}_{0k} U_0$, we have:

$$\langle \mathbf{D}_0 U_0, U_0 \rangle \leq \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k V_k, V_k \rangle . \qquad (21)$$

For all $k \geq 1$, we decompose V_k into $V_k = U_k^* + \delta U_k$, where (U_k^*, Λ_k^*) denotes the solution of (15), and $\mathbf{B}_k \delta U_k = 0$. One has $\langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k U_k^*, U_k^* \rangle = \langle \mathbf{D}_0 U_0, U_0 \rangle$, and by symmetry of $\mathbf{A}_k$

$$\langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k V_k, V_k \rangle = \langle \mathbf{D}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} 2 \langle \mathbf{A}_k U_k^*, \delta U_k \rangle + \langle \mathbf{A}_k \delta U_k, \delta U_k \rangle .$$

Furthermore, $\langle \mathbf{A}_k U_k^*, \delta U_k \rangle = \left\langle \mathbf{B}_k^\top \Lambda_k^*, \delta U_k \right\rangle = \langle \Lambda_k^*, \mathbf{B}_k \delta U_k \rangle = 0$ which implies

$$\begin{aligned} \langle \mathbf{A}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k V_k, V_k \rangle &= \langle \mathbf{D}_0 U_0, U_0 \rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k \delta U_k, \delta U_k \rangle \\ &\geq \langle \mathbf{D}_0 U_0, U_0 \rangle . \end{aligned}$$

Hence (21) which completes the claim. □

4.3 Spectral equivalence for the enhanced Dirichlet Neumann

For the enhanced Dirichlet-Neumann preconditioner presented in section 3.3.2, we prove that:

Proposition 3 *For all U_0, the following spectral equivalence holds:*

$$W_{1,h} \langle \mathbf{D}_0 U_0, U_0 \rangle \leq \left\langle \hat{\mathbf{D}}_0 U_0, U_0 \right\rangle \leq \langle \mathbf{D}_0 U_0, U_0 \rangle ,$$

with

$$\frac{1}{W_{1,h}} = C \left(1 + \max_{k \in I_1 \cup I_2} \frac{C_k}{c_0} \right),$$

where I_1 (resp. I_2) is the set of indices $k \geq 1$ such that Ω_k is not fixed on its boundary (resp. is fixed on a part of its boundary). The constant C is independent of the number K and the size of the subdomains.

Proof : Let U_0 be given. We proceed as in the last part of the previous proof, and introduce (U_k^*, Λ_k^*) satisfying (15). We introduce the decomposition $U_k^* = \mathring{U}_k^* + W_k^*$ with $\mathring{U}_k^* = \Pi_k U_0$, and by construction of U_k^*, we obtain

$$
\begin{aligned}
\langle \mathbf{D}_0 U_0, U_0\rangle &= \left\langle \left(\mathbf{A}_0 - \sum_{k=1}^{K} \mathbf{B}_{0k}^{\top} R_k \mathbf{K}_k^{-1} R_k^{\top} \mathbf{B}_{0k} \right) U_0, U_0 \right\rangle \\
&= \langle \mathbf{A}_0 U_0, U_0\rangle + \sum_{k=1}^{K} \langle \mathbf{A}_k U_k^*, U_k^*\rangle \\
&\geq \langle \mathbf{A}_0 U_0, U_0\rangle + \sum_{k=1}^{K} \left\langle \mathbf{A}_k \mathring{U}_k^*, \mathring{U}_k^* \right\rangle + 2 \left\langle \mathbf{A}_k \mathring{U}_k^*, W_k^* \right\rangle .
\end{aligned} \tag{22}
$$

Exploiting the decomposition $\mathring{U}_k^* = \mathring{\Pi}_k U_0 = \sum_{j=1}^{N_k} z_j U_k^j$ provides

$$
\begin{aligned}
\left\langle \mathbf{A}_k W_k^*, \mathring{U}_k^* \right\rangle &= \sum_{j=1}^{N_k} z_j a_k(w_k^*, u_k^j) = \sum_{j=1}^{N_k} z_j \int_{\Gamma_{0k}} \lambda_k^j \cdot w_k^*, \quad \text{from (8).1,} \\
&= \sum_{j=1}^{N_k} z_j \int_{\Gamma_{0k}} (u_k^* - \mathring{u}_k^*) \cdot \lambda_k^j, \quad \text{by construction of } w_k^*, \\
&= \sum_{j=1}^{N_k} z_j \left[\int_{\Gamma_{0k}} u_k^* \cdot \lambda_k^j - \int_{\Gamma_{0k}} \mathring{u}_k^* \cdot \lambda_k^j \right], \quad \text{from (15) and (12).2,} \\
&= 0.
\end{aligned}
$$

Plugging the latter result into (22) gives

$$
\begin{aligned}
\langle \mathbf{D}_0 U_0, U_0\rangle &\geq \langle \mathbf{A}_0 U_0, U_0\rangle + \sum_{k=1}^{K} \left\langle \mathbf{A}_k \mathring{U}_k^*, \mathring{U}_k^* \right\rangle \\
&= \langle \mathbf{A}_0 U_0, U_0\rangle + \sum_{k=1}^{K} \left\langle \mathbf{A}_k \mathring{\Pi}_k U_0, \mathring{\Pi}_k U_0 \right\rangle \\
&= \left\langle \left(\mathbf{A}_0 + \sum_{k=1}^{K} \mathring{\Pi}_k^{\top} \mathbf{A}_k \mathring{\Pi}_k \right) U_0, U_0 \right\rangle \\
&= \langle \mathbf{A}_0 U_0, U_0\rangle + \sum_{k=1}^{K} \left\langle \mathbf{A}_k \mathring{U}_k^*, \mathring{U}_k^* \right\rangle \quad \text{from (13),} \\
&= \left\langle \hat{\mathbf{D}}_0 U_0, U_0 \right\rangle .
\end{aligned}
$$

Let us prove now a lower bound for $\hat{\mathbf{D}}_0$. For U_0 given and for all $1 \leq k \leq K$, as in the proof of the previous proposition, we build a particular function $u_k \in W_{k;\delta_k}$ satisfying the weak continuity constraint on the interface Γ_{0k}. When Ω_k is not fixed on a part of its boundary, which we have denoted by $k \in I_1$, we take the u_k defined in the previous proof by "reflexion" with respect to Γ_{0k}. When Ω_k is fixed on a part of its boundary, namely $k \in I_2$, we proceed differently, and define here $\langle u_0 \rangle_k \in \mathcal{R}(\Gamma_{0k})$ (the trace over Γ_{0k} of a rigid motion) such that

$$\int_{\Gamma_{0k}} \langle u_0 \rangle_k \cdot r = \int_{\Gamma_{0k}} u_0 \cdot r, \quad \forall r \in \mathcal{R}(\Gamma_{0k}).$$

Then, we introduce $u_k = \mathcal{I}_{k;h_k}\tilde{u}_k + \mathcal{R}_{k;\delta_k}\pi_k\left[\tilde{u}_k - \mathcal{I}_{k;h_k}\tilde{u}_k\right] + \mathring{\pi}_k \langle u_0 \rangle_k$, where $\tilde{u}_k$ is a function whose trace is zero on Γ_D and is $u_0 - \langle u_0 \rangle_k$ on Γ_{0k} satisfying from the Sobolev lifting theorem the following estimate

$$\begin{aligned}\int_{\Omega_k} |\nabla \tilde{u}_k|^2 &\leq C\left[\frac{1}{L_k}\left\langle u_0 - \langle u_0\rangle_k\right\rangle_k + |u_0 - \langle u_0\rangle_k|^2_{H^{1/2}(\Gamma_{0k})^d}\right] \\ &= C|u_0 - \langle u_0\rangle_k|^2_{H^{1/2}(\Gamma_{0k})^d}, \quad \text{by construction of } \langle u_0\rangle_k. \qquad (23)\end{aligned}$$

The mortar condition is indeed satisfied because

$$\begin{aligned}\int_{\Gamma_{0k}} u_k \cdot \mu &= \int_{\Gamma_{0k}} (\mathcal{I}_{k;h_k}\tilde{u}_k + \tilde{u}_k - \mathcal{I}_{k;h_k}\tilde{u}_k)\cdot\mu + \int_{\Gamma_{0k}} \mathring{\pi}_k \langle u_0\rangle_k \cdot \mu \\ &= \int_{\Gamma_{0k}} \tilde{u}_k \cdot \mu + \int_{\Gamma_{0k}} \mathring{\pi}_k \langle u_0\rangle_k \cdot \mu \\ &= \int_{\Gamma_{0k}} (u_0 - \langle u_0\rangle_k + \mathring{\pi}_k \langle u_0\rangle_k)\cdot\mu, \quad \forall \mu \in M_{k;\delta_k},\end{aligned}$$

and because, since $\langle u_0\rangle_k$ is a linear combination of rigid body motions e_k^i, we have from (14)

$$\int_{\Gamma_{0k}} (\langle u_0\rangle_k - \mathring{\pi}_k \langle u_0\rangle_k)\cdot\mu = 0, \quad \forall \mu \in M_{k;\delta_k}.$$

On the other hand, we have for $k \in I_2$

$$\begin{aligned}a_k(u_k, u_k) &\leq 2a_k\left(u_k - \mathring{\pi}_k \langle u_0\rangle_k, u_k - \mathring{\pi}_k \langle u_0\rangle_k\right) \\ &\quad +2a_k\left(\mathring{\pi}_k \langle u_0\rangle_k, \mathring{\pi}_k \langle u_0\rangle_k\right). \qquad (24)\end{aligned}$$

Using the same argument as in (17), we get by construction of u_k

$$\begin{aligned}a_k\left(u_k - \mathring{\pi}_k \langle u_0\rangle_k, u_k - \mathring{\pi}_k \langle u_0\rangle_k\right) &\leq CC_k \int_{\Omega_k} |\nabla\tilde{u}_k|^2 \\ &\leq CC_k\,|u_0 - \langle u_0\rangle_k|^2_{H^{1/2}(\Gamma_{0k})^d}, \quad \text{from (23)}, \\ &\leq CC_k \int_{\Omega'_k} \varepsilon(u_0) : \varepsilon(u_0), \qquad (25)\end{aligned}$$

from the Sobolev trace theorem and the inequality (18). On the other hand, we have from lemma 2

$$
\begin{aligned}
a_k\left(\mathring{\pi}_k \left\langle u_0\right\rangle_k, \mathring{\pi}_k \left\langle u_0\right\rangle_k\right) \leq & 2a_k\left(\mathring{\pi}_k\left(u_0-\left\langle u_0\right\rangle_k\right), \mathring{\pi}_k\left(u_0-\left\langle u_0\right\rangle_k\right)\right)\\
& +2a_k\left(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0\right)\\
\leq & C C_k \left|u_0-\left\langle u_0\right\rangle_k\right|^2_{H^{1/2}(\Gamma_{0k})^d} + 2a_k\left(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0\right)\\
\leq & C C_k \int_{\Omega'_k} \varepsilon(u_0):\varepsilon(u_0) + 2a_k\left(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0\right).
\end{aligned}
$$

Equations (19),(24) and (25) enable to obtain

$$
\begin{aligned}
a_0(u_0,u_0)+\sum_{k=1}^{K} a_k(u_k,u_k) \leq & a_0(u_0,u_0) + C\sum_{k=1}^{K} C_k \int_{\Omega'_k} \varepsilon(u_0):\varepsilon(u_0)\\
& +4a_k\left(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0\right)\\
\leq & \left(4+\frac{C}{c_0}\max_{k\geq 1}(C_k)\right)\left[a_0(u_0,u_0)+\sum_{k\in I_2} a_k\left(\mathring{\pi}_k u_0, \mathring{\pi}_k u_0\right)\right]\\
= & \left(4+\frac{C}{c_0}\max_{k\geq 1}(C_k)\right)\left\langle \hat{\mathbf{D}}_0 U_0, U_0\right\rangle.
\end{aligned}
$$

Finally, from (21) and the mortar conditions satisfied by the $(u_k)_{k\geq 1}$, the following upper bound holds

$$
\langle \mathbf{D}_0 U_0, U_0\rangle \leq \left(4+\frac{C}{c_0}\max_{k\geq 1}(C_k)\right)\left\langle \hat{\mathbf{D}}_0 U_0, U_0\right\rangle.
$$

□

In the above proof, we have used the following lemma:

Lemma 2 *If a_k is symmetric, the projection operator $\mathring{\pi}_k$ satisfies:*

$$
a_k\left(\mathring{\pi}_k w, \mathring{\pi}_k w\right) \leq C C_k \left[\frac{1}{L_k}\int_{\Gamma_{0k}} \langle w\rangle_k^2 + |w|^2_{H^{1/2}(\Gamma_{0k})^d}\right]
$$

Proof : Let $\tilde{w}$ be a lifting function of w with zero trace on Γ_D, with $\tilde{w}=w$ on Γ_{0k} and satisfying the Sobolev lifting theorem

$$
\int_{\Omega_k} |\nabla \tilde{w}|^2 \leq C\left[\frac{1}{L_k}\int_{\Gamma_{0k}} \langle w\rangle_k^2 + |w|^2_{H^{1/2}(\Gamma_{0k})^d}\right].
$$

Let us define as before $\tilde{w}_k = \mathcal{I}_{k;h_k}\tilde{w} + \mathcal{R}_{k;\delta_k}\pi_k(\tilde{w}-\mathcal{I}_{k;h_k}\tilde{w})$ which belongs to $X_{k;h_k}$ and which satisfies by construction

$$
\int_{\Gamma_{0k}} \tilde{w}_k\cdot\mu = \int_{\Gamma_{0k}} \tilde{w}\cdot\mu, \quad \forall \mu\in M_{k;\delta_k}. \tag{26}
$$

We then have on one hand

$$
\begin{aligned}
a_k(\tilde{w}_k,\tilde{w}_k) = & a_k(\mathring{\pi}_k w, \mathring{\pi}_k w) + a_k(\mathring{\pi}_k w - \tilde{w}_k, \mathring{\pi}_k w - \tilde{w}_k)\\
& +2a_k(\mathring{\pi}_k w, \mathring{\pi}_k w - \tilde{w}_k).
\end{aligned} \tag{27}
$$

Developing $\mathring{\pi}_k w_k$ into $\mathring{\pi}_k w_k = \sum_{j=1}^{N_k} z_j u_k^j$, we have from (8).1

$$\begin{aligned}
& a_k(\mathring{\pi}_k w_k - \tilde{w}_k, \mathring{\pi}_k w_k) \\
=& \sum_{j=1}^{N_k} z_j a_k(\mathring{\pi}_k w - \tilde{w}_k, u_k^j) = \sum_{j=1}^{N_k} z_j \int_{\Gamma_{0k}} (\mathring{\pi}_k w - \tilde{w}_k) \cdot \lambda_k^j \\
=& \sum_{j=1}^{N_k} z_j \left[\int_{\Gamma_{0k}} w \cdot \lambda_k^j - \int_{\Gamma_{0k}} w \cdot \lambda_k^j \right], \quad \text{from (12).2 and (26)} \\
=& \, 0.
\end{aligned}$$

Plugged back in (27), this implies $a_k(\mathring{\pi}_k w, \mathring{\pi}_k w) \leq a_k(\tilde{w}_k, \tilde{w}_k)$. But on the other hand, proceeding as in (17), we get

$$a_k(\tilde{w}_k, \tilde{w}_k) \leq CC_k \int_{\Omega_k} |\nabla \tilde{w}|^2 \leq CC_k \left[\frac{1}{L_k} \int_{\Gamma_{0k}} \langle w \rangle_k^2 + |w|^2_{H^{1/2}(\Gamma_{0k})^d} \right]$$

the last inequality coming from the Sobolev lifting theorem. This concludes the proof. □

4.3.1 Bound on condition number

Referring to the Matsokin-Nepomniaschik framework [MN85], the spectral equivalence proved for our preconditioners in the previous sections implies the main result of the section:

Proposition 4 *For the symmetrized Dirichlet-Neumann preconditioner given in section 3.3.1, we have*

$$\kappa_{\mathbf{A},E}(\mathbf{C}^{-1}\mathbf{A}) \leq 1 + C \left(\max_{k \in I_1} \frac{C_k}{c_0} + \max_{k \in I_2} \frac{C_k L_0}{\alpha_0 L_k} \right),$$

and for the enhanced Dirichlet-Neumann preconditioner given in section 3.3.2

$$\kappa_{\mathbf{A},E}(\mathbf{C}^{-1}\mathbf{A}) \leq C \left(1 + \max_{k \in I_1 \cup I_2} \frac{C_k}{c_0} \right).$$

Both condition numbers are independent of the number K of fine scale subdomains and of their sizes. In that sense, we can reasonably talk of two-scale preconditioners. The simplest symmetrized Dirichlet-Neumann preconditioner, which imposes the invertibility of $\mathbf{A}_0$ (i.e. a Dirichlet boundary condition on Ω_0 for example), is strongly affected by the presence of small subdomains that are fixed on a part of their boundary, through the ratio L_0/L_k. The enhanced symmetrized Dirichlet-Neumann preconditioner avoids efficiently this dependence, and its use is not limited to the case where $\Gamma_D \cap \partial\Omega_0$ has a positive measure. Nevertheless, both condition numbers are affected by the presence of stiff fine subdomains in comparison with the coarse domain, through the presence of the ratio C_k/α_0 because C_k (resp. α_0) is proportional to the Young modulus E_k (resp E_0) of the material in Ω_k (resp. Ω_0). We acknowledge that this limitation is a natural limitation of Dirichlet-Neumann preconditioners.

5 Algorithm

Before testing these two preconditioners, we summarize herein the way of implementing the associated algorithms. The action of a preconditioner on a right hand side $(F_0, F_1, ..., F_K)^\top$ in the dual of E leads to the following sequence of operations :

(1) Compute the equivalent coarse scale sollicitation on Ω_0

$$\overline{F_0} = F_0 - \sum_{k=1}^{K} \mathbf{B}_{0k}^\top R_k \mathbf{K}_k^{-1} \begin{pmatrix} F_k \\ 0 \end{pmatrix},$$

by solving in parallel one Dirichlet problem by small subdomain. On each subdomain, the force F_k is imposed and a weak zero trace prescribed on the interface.

(2) Use the equivalent coarse scale operator $\hat{\mathbf{D}}_0$ to determine $\tilde{U}_0 = \hat{\mathbf{D}}_0^{-1}\overline{F_0}$.

(3) Solve the local problems for $1 \leq k \leq K$

$$\mathbf{K}_k \begin{pmatrix} \tilde{U}_k \\ \tilde{\Lambda}_k \end{pmatrix} = \begin{pmatrix} F_k \\ -\mathbf{B}_0 \tilde{U}_0 \end{pmatrix}.$$

On each subdomain, the force F_k is imposed and $\tilde{U}_0$ is prescribed in the sense of weak traces on the interface.

If the computational cost of $\mathbf{A}_k^{-1}$ for $k \geq 1$ is low with respect to the one of $\mathbf{A}_0^{-1}$, the computation cost is concentrated in the step 2. In this situation, the cost of computation of the coupled problem is comparable to the cost of computation of the coarse problem posed on Ω_0.

The proposed preconditioners are multiplicative in the sense that the two scales cannot be solved simultaneously. Nevertheless, the solutions over the small details can be performed in parallel.

6 Numerical tests

6.1 A basic two-scale model

Let us consider a two-scale linear model two-dimensional beam whose tips are clamped. We impose a negative constant pressure on the lower face of the small sculptures. A $\mathbb{Q}_1$ approximation is adopted for displacements, and an example of the resulting deformed configuration of our model is represented on figure 3. The Young modulus and the Poisson coefficient are taken constant over the coarse (E_0, ν_0) and the fine (E', ν') subdomains. As assumed above, the non-mortar side is taken as the fine side of the interface and Lagrange multipliers are taken piecewise constant, together with an interface bubble stabilization for the displacements (see [Hau04]). Moreover, the weak-continuity constraint is ensured by a penalization strategy and the associated penalization coefficient is taken as $10^6 E'$.

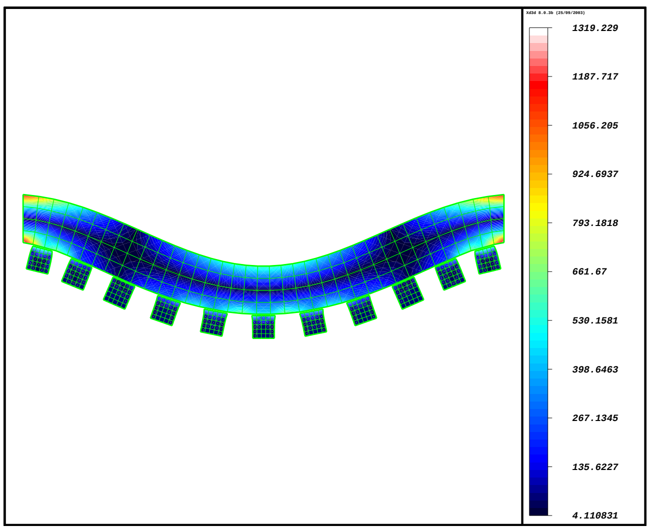

Fig. 3. Maximal stress distribution on a deformed configuration of our two-scale model problem ($E_0 = E'$, $\nu_0 = \nu'$, 497 elements mesh). The non-conformity ratio (i.e. number of fine/coarse elements) on interfaces is 3.3.

On this model, we use the first symmetrized Dirichlet-Neumann preconditioner in a standard Conjugate Gradient algorithm. Figure 4 describes the evolution of the L^2 norm of the successive increments of Lagrange multipliers during iterations for different values of the ratio $r = E'/E_0$. Conversely the number of iterations necessary to obtain a 10^{-9} convergence, estimated in terms of the L^2 norm of the current increment on the Lagrange multiplier, is shown on figure 5 as a function of the ration $r = E'/E_0$. The degradation of the performance as r grows is in conformity with our predictions.

Let us assume now, that two of the details are clamped on their lower face,

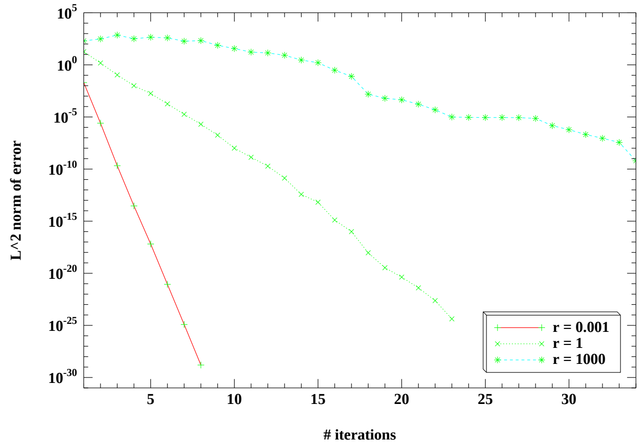

Fig. 4. L^2 norm of the successive increments on Lagrange multipliers along the iterations.

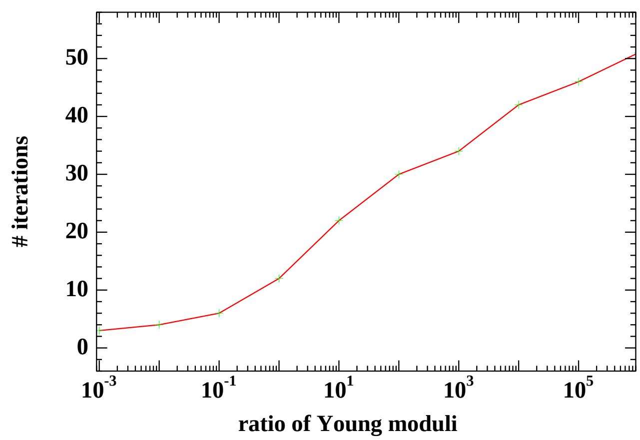

Fig. 5. Number of iterations necessary to obtain a 10^{-9} convergence of the simple Dirichlet-Neumann preconditioned Conjugate Gradient, estimated in terms of the L^2 norm of the current increment on the Lagrange multiplier, as a function of the ratio $r = E'/E_0$.

leading under the same load to the new deformed configuration illustrated on figure 6. The convergence of simple and enhanced Dirichlet-Neumann algorithms are then compared on figures 7 and 8 for the ratios $r = 1$, 10, 100, 1000, 10^6. Conversely, the number of iterations necessary to reach a 10^{-9} convergence as a function of r is represented on figure 9 both for simple and enhanced Dirichlet-Neumann algorithms. We observe a much better performance of the enhanced preconditioner, the number of iterations being typically divided by 3 for an additional computational cost of 6 additional degrees of freedom on the coarse part of the model. Indeed, 3 rigid motions per clamped small structure have been added to the coarse model. The resulting overcost per iteration in terms of computation is negligible.

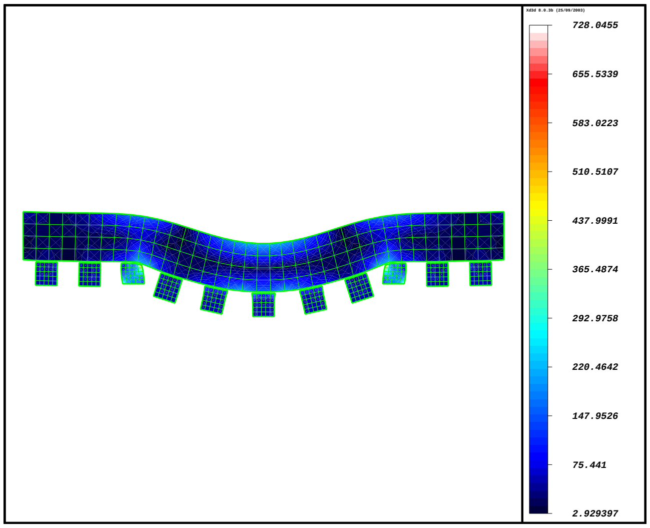

Fig. 6. Maximal stress distribution on a deformed configuration of our two-scale model problem where two of the details are clamped on their lower face ($E_0 = E'$, $\nu_0 = \nu'$, 497 elements mesh).

6.2 Extension to a quasi-Newton method

When considering nonlinear problems with soft fine geometrical details on the boundary, the previous preconditioners can be successfully applied to quasi-Newton methods. More precisely, we propose to replace in a standard Newton-Raphson method the tangent problems by the approximation provided by the preconditioner.

The linear law of the previous sections is replaced by a Saint Venant-Kirchhoff material characterized by the hyperelastic energy

$$\hat{\mathcal{W}}(F) = \frac{\lambda}{4}\left[tr(F^\top \cdot F - id)\right]^2 + \frac{\mu}{8}\, tr\left[(F^\top \cdot F - id)^2\right],$$

in which $F = Id + \nabla u$. Let us consider the unclamped model problem (see figure 3) of the previous section, under a dead pressure of $p = 100Pa$. We have used the following Lamé coefficients

$$\lambda_0 = E_0 \frac{\nu_0}{(1+\nu_0)(1-2\nu_0)} = 1389Pa, \quad \mu_0 = \frac{E_0}{2(1+\nu_0)} = 2083Pa,$$

$$\lambda' = r\lambda_0, \qquad \mu' = r\mu_0 Pa,$$

respectively for the coarse and the fine subdomains, characterized by the stiffness ratio

$$r = \frac{E_0}{E'} = \frac{\lambda_0}{\lambda'} = \frac{\mu_0}{\mu'}.$$

We have observed numerically that for $r \geq 10$, the quasi-Newton method does

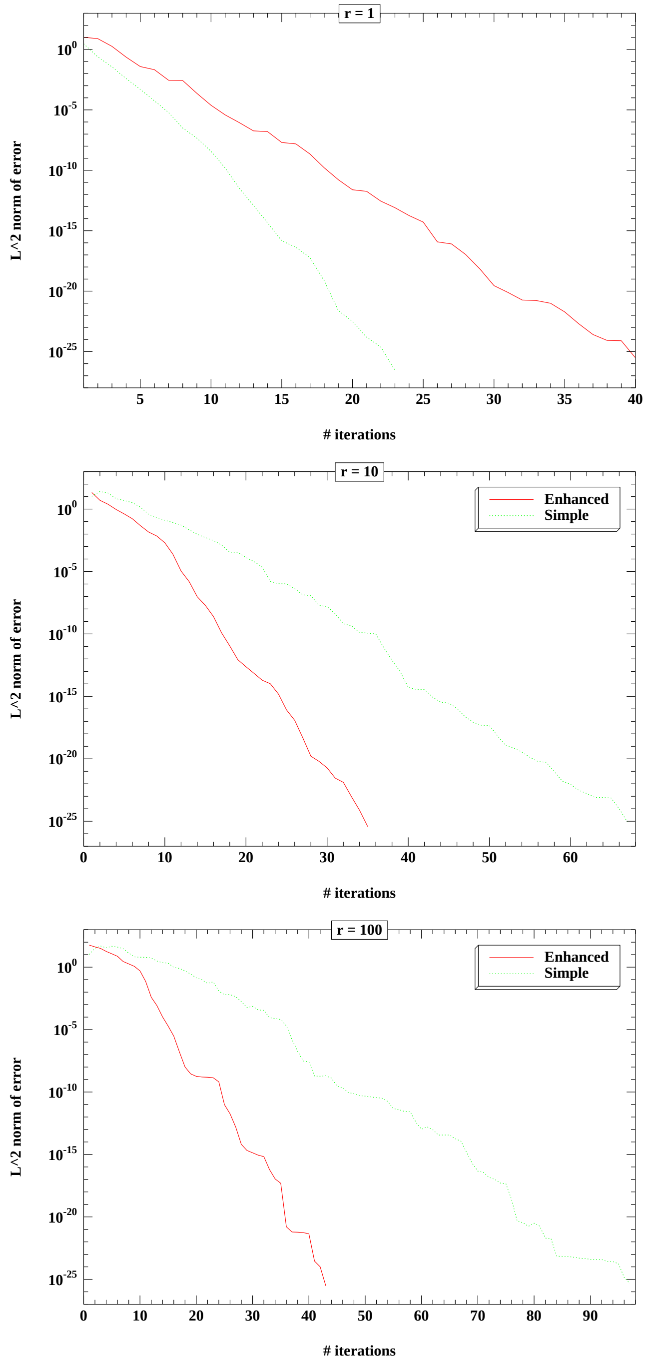

Fig. 7. Convergence of the simple and enhanced Dirichlet-Neumann algorithms for different values of the ratio r of Young moduli; $r = 1, 10, 100$.

not converge well, as shown on the table, figure 10. The convergence remains slow with $r = 1$; it explodes for $r = 100$. The convergence of the Newton-Raphson method is represented as a comparison. Nevertheless, when the ratio r remains sufficiently small, the proposed quasi-Newton method appears to be interesting, even though the convergence is no more quadratic. The overcost in terms of iterations compared with a Newton-Raphson method is low, as shown

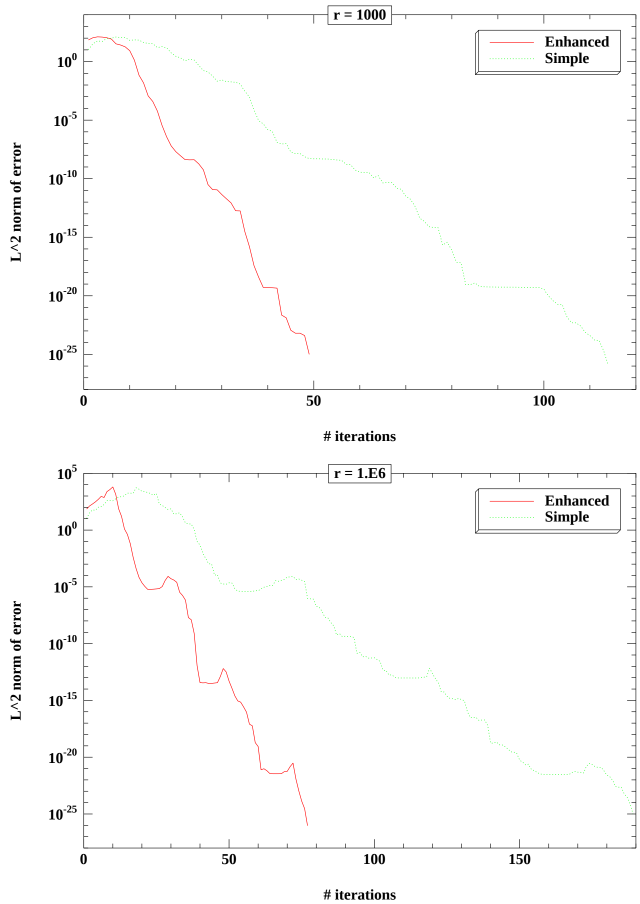

Fig. 8. Convergence of the simple and enhanced Dirichlet-Neumann algorithms for different values of the ratio r of Young moduli; $r = 1000, 10^6$.

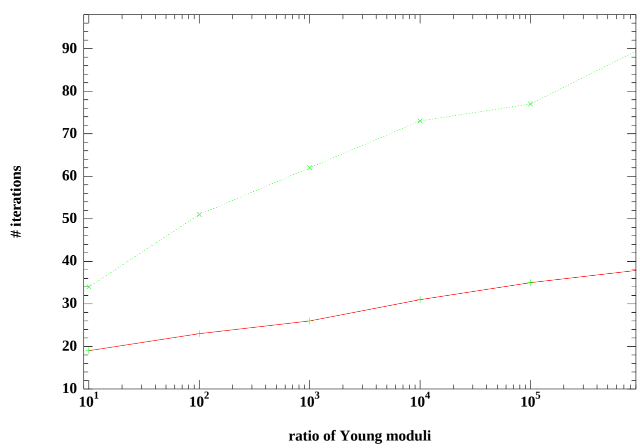

Fig. 9. Number of iterations necessary to obtain a 10^{-9} convergence of the simple and the enhanced Dirichlet-Neumann preconditioned Conjugate Gradient, estimated in terms of the L^2 norm of the current increment on the Lagrange multiplier, as a function of the ratio $r = E'/E_0$.

in the table, figure 11. Finally, we represent on figure 12 the different evolutions of the L^2 norm of the residual for the proposed quasi-Newton method along the iterations, depending on the value of the ratio r.

	$r = 1$		$r = 100$	
it.	quasi-Newton	Newton	quasi-Newton	Newton
1	0.6193E+01	0.5839E+01	0.6187E+01	0.5224E+01
2	0.1904E+01	0.1649E+01	0.1380E+02	0.1401E+01
3	0.1013E+01	0.9821E+00	0.6958E+03	0.7683E+00
4	0.6684E+00	0.6221E+00	0.1283E+04	0.4046E+00
5	0.3309E+00	0.3032E+00	0.3672E+04	0.2419E+00
6	0.8885E-01	0.8811E-01	0.1847E+04	0.1454E+00
7	0.4654E-02	0.8719E-02	0.9162E+03	0.1096E+00
8	0.5162E-02	0.1591E-03	0.6159E+03	0.5302E-01
9	0.4352E-02	0.8287E-07	0.1027E+04	0.5350E-01
10	0.3714E-02		0.6719E+03	0.7019E-02
11	0.3155E-02		0.8720E+03	0.3277E-02
12	0.2716E-02		0.5561E+03	0.3023E-04
13	0.2334E-02		0.6285E+03	0.3357E-07
14	0.2023E-02		0.8873E+03	
15	0.1753E-02		0.5120E+03	
16	0.1528E-02		0.5499E+03	
17	0.1333E-02		0.6496E+03	
18	0.1167E-02		0.9376E+03	
19	0.1023E-02		0.3581E+03	
20	0.8981E-03		0.3805E+03	
21	0.7895E-03		0.5372E+03	
22	0.6950E-03		0.8865E+03	
23	0.6123E-03		0.7312E+03	
24	0.5399E-03		0.7739E+03	
25	0.4764E-03		0.7279E+03	

Fig. 10. Slow convergence of the method for $r = 1$, and lack of convergence for $r = 100$.

	L^2 norm of the residual with	
it.	Newton algorithm	two-scale quasi-Newton
1	0.6192E+01	0.6249E+01
2	0.1775E+01	0.1811E+01
3	0.1061E+01	0.1075E+01
4	0.6671E+00	0.6747E+00
5	0.3254E+00	0.3292E+00
6	0.8096E-01	0.7836E-01
7	0.5036E-02	0.3414E-02
8	0.2010E-04	0.8871E-05
9	0.3750E-09	0.5000E-06
10	converged	0.1387E-07
11	converged	0.3629E-08

Fig. 11. Convergence of the exact Newton and two-scale quasi-Newton algorithm using the simple preconditioner. We have chosen $E_0/E' = 10$ and the convergence criterion is that the L^2 norm of the residual become $\leq 10^{-9}$.

7 Conclusion

In this paper, we have analyzed and tested two symmetrized Dirichlet-Neumann preconditioners that can be used efficiently together with a non-conforming mortar formulation to solve elliptic problems with small geometrical details on

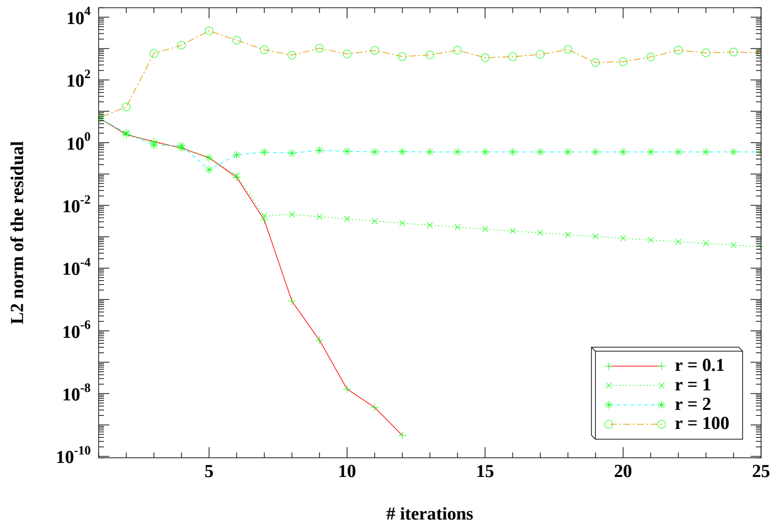

Fig. 12. Evolutions of the L^2 norm of the residual for the proposed quasi-Newton method along the iterations, depending on the value of the ratio r.

the boundary. This method is well-adapted to the case where the details are localized enough to make their resolution relatively cheap. In the case where the small structures would not be so localized to satisfy this assumption, one could imagine another level of hierarchy in the preconditioner through which the Dirichlet problem would be split. Finally, we have deduced a quasi-Newton method which is well suited for soft details in the framework of nonlinear problems.

References

[AAKP99] G. Abdulaiev, Y. Achdou, Y. Kuznetsov, and C. Prudhomme. On a parallel implementation of the mortar element method. *M2AN*, 33(2), 1999.

[ABFI99] M. Azaiez, F. Ben Belgacem, H. El Fekih, and M. Ismael. Numerical simulation of the wave equation with discontinuous coefficients by nonconforming finite elements. *Numer. Methods Partial Differ. Equations*, 15(6):637–656, 1999.

[AKP95] Y. Achdou, Y. Kuznetsov, and O. Pironneau. Substructuring preconditioners for the Q_1 mortar element method. *Numerische Mathematik*, 71:419–449, 1995.

[AMW99] Y. Achdou, Y. Maday, and O. Widlund. Substructuring preconditioners for the mortar method in dimension two. *SIAM Journal of Numerical Analysis*, 32(2):551–580, 1999.

[AT95] A. Agouzal and J.M. Thomas. Une méthode d'éléments finis hybrides en décomposition de domaines. *RAIRO M2AN*, 29:749–764, 1995.

[Bel99] F. Ben Belgacem. The mortar finite element method with lagrange multipliers. *Numer. Math.*, 84:173–197, 1999.

[Bel04] F. Ben Belgacem. A stabilized domain decomposition method with non-matching grids for the stokes problem in three dimensions. *SIAM Journal of Numerical Analysis*, 42(2):667–685, 2004.

[BM00] F. Brezzi and D. Marini. Error estimates for the three-field formulation with bubble stabilization. *Math. Comp.*, 70:911–934, 2000.

[BMP93] C. Bernardi, Y. Maday, and A.T. Patera. Domain decomposition by the mortar element method. In H.G. Kaper ans M. Garbey, editor, *Asymptotic and Numerical Methods for Partial Differential Equations with Critical Parameters*, pages 269–286. N.A.T.O. ASI, Kluwer Academic Publishers, 1993.

[BMP94] C. Bernardi, Y. Maday, and A.T. Patera. *Nonlinear partial differential equations and their applications.*, chapter A new nonconforming approach to domain decomposition: the mortar element method., pages 13–51. Pitman, Paris, 1994.

[Bré99] H. Brézis. *Analyse fonctionnelle.* Dunod, 1999.

[Bre04] S. Brenner. Korn's inequalities for piecewise H^1 vector fields. *Mathematics of Computation*, 73:1067–1087, 2004.

[DL72] G. Duvaut and J-L. Lions. *Les inéquations en Mécanique et en Physique.* Dunod, 1972.

[FR91] C. Farhat and F-X. Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Int. J. Num. Meth. Engr.*, 32:1205–1228, october 1991.

[Hau04] P. Hauret. *Méthodes numériques pour la dynamique des structures non-linéaires incompressibles à deux échelles (Numerical methods for the dynamic analysis of two-scale incompressible nonlinear structures).* PhD thesis, Ecole Polytechnique, 2004.

[HT04a] P. Hauret and P. Le Tallec. A stabilized discontinuous mortar formulation for elastostatics and elastodynamics problems, part i: abstract framework. Technical Report 553, CMAP, september 2004.

[HT04b] P. Hauret and P. Le Tallec. A stabilized discontinuous mortar formulation for elastostatics and elastodynamics problems, part ii: discontinuous lagrange multipliers. Technical Report 554, CMAP, september 2004.

[MN85] A. Matsokin and S. Nepomniaschik. A Schwarz alternating method in a subspace. *Sov. Math.*, 29:78–84, 1985.

[QV99] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations.* Oxford University Press, 1999.

[Ses98] P. Seshaiyer. *Non-conforming hp finite element methods.* PhD thesis, University of Maryland, 1998.

[Ste99] D. Stefanica. *Domain decomposition methods for mortar finite elements.* PhD thesis, Courant Institute of Mathematical Sciences, New York University, 1999.

[SZ90] L.R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, april 1990.

[Tal93] P. Le Tallec. Neumann-Neumann domain decomposition algorithm for solving 2d elliptic problems with nonmatching grids. *East-West J. Numer. Math.*, 1(2):129–146, 1993.

[TRV91] P. Le Tallec, Y.H. De Roeck, and M. Vidrascu. Domain decomposition methods for large linearly elliptic three dimensional problems. *J. Comput. Appl. Math.*, 34:93–117, 1991.

[Woh99] B.I. Wohlmuth. Hierarchical a posteriori error estimators for mortar finite element methods with lagrange multipliers. *SIAM J. Numer. Anal.*, 36:1636–1658, 1999.

[Woh00] B.I. Wohlmuth. A mortar finite element method using dual spaces for the lagrange multiplier. *SIAM J. Numer. Anal.*, 38:989–1012, 2000.

[Woh01] B.I. Wohlmuth. *Discretization methods and iterative solvers based on domain decomposition.* Springer, 2001.